Thrust from Arcjets with Self-Induced Magnetic Fields

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Using the quasi one-dimensional magnetohydrodynamic equations, a new formula for the thrust from arcjets with self-induced magnetic fields is derived. The theory takes into consideration the nonlinear interactions between the current and flowfield, and thus the resulting thrust formula is not the sum of the various components. If the plasma behaves like a perfect gas, then closed-form expressions for the various performance parameters can be derived. It is shown that the exhaust velocity increases with a decreasing mass flow rate. Moreover, the square of the exhaust velocity, for given arc conditions, is approximately a linear function of the square of the current. The results of the theory are employed to suggest methods for predicting electrode erosion and/or entrainment.

Introduction

EXPERIMENTS on high current steady,[1–5] and quasi-steady[6–9] MPD arcs with self-induced magnetic fields raised a number of questions regarding the possible acceleration mechanisms in these devices. The nature of these questions can be traced to the thrust formula that assumes the thrust to be the sum of the aerodynamic and electromagnetic contributions. Thus, it was evident from the beginning that the usual formula for the thrust coefficient, which yields a value of about 1.5, was not valid. Earlier work[1] suggested a value of unity; however, more recent experiments[2–5] indicated that even such value of the thrust coefficient was unacceptable. Another problem associated with the thrust formula was uncovered by experiments on quasi-steady devices.[7–9] One expects that for the high currents employed, the thrust will be purely electromagnetic and, therefore, is independent of the mass flow rate. However, it was found that the exit velocity, and not the thrust, was almost independent of the mass flow rate below a certain metered mass flow rate.

Because interactions between the current and flowfield are, in general, nonlinear, any theory that attempts to explain the acceleration mechanism in high current arcjets must take into consideration such nonlinear interactions. Attempts directed toward redefining the thrust coefficient may help in interpreting some experimental data; however, because they are empirical in nature they are not expected to clarify the acceleration mechanism. What is needed is a new theory which takes into consideration the nonlinear interaction between the current and flowfield and which is not built around the usual simple thrust formula.[1–5] This is the object of this investigation.

The analysis, even in the absence of erosion and/or entrainment, is extremely complicated. Because of the amount of computations involved and because of the lack of precise measurements, the usual lengthy numerical computations are not justified at present. Instead, some simplifying assumptions are introduced which make it possible to derive an explicit formula for the thrust. Thus, the quasi one-dimensional magnetohydrodynamic equations[10] are used as a basis for the model. In addition, it is assumed that the magnetic Reynolds number downstream of the arc chamber is large so that the flow in that region may be assumed isentropic. Because chemical and thermal nonequilibrium effects are not taken into consideration and because of the isentropic approximation downstream of the arc chamber, the model employed here is an oversimplification of the actual situation. However, because it takes into consideration the nonlinear interaction between the current and flowfield and because the thrust is an over-all property of the arcjet, it is expected that the trends predicted by the model will not be much influenced by the assumptions introduced here.

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Analysis

Using the magnetohydrodynamic approximation, the equations governing the plasma at high magnetic Reynolds numbers are the conservation equations of mass and momentum, the induction equation, and the statement that the entropy is constant. These equations can be written as[10]

$$D\rho/Dt + \rho\nabla\cdot\mathbf{u} = 0 \tag{1}$$

$$\rho(D\mathbf{u}/Dt) + \nabla p = (\nabla \times \mathbf{B}) \times \mathbf{B}/\mu \tag{2}$$

$$(D\mathbf{B}/Dt) + \mathbf{B}(\nabla\cdot\mathbf{u}) = (\mathbf{B}\cdot\nabla)\mathbf{u} \tag{3}$$

where ρ is the density, p the pressure, $\mathbf{u}$ the mean velocity, $\mathbf{B}$ the magnetic induction, and μ the permeability of free space. For an axisymmetric MPD arc with self-induced fields,

$$\mathbf{B} = \mathbf{i}_\theta B \tag{4}$$

and therefore the operator,

$$\mathbf{B}\cdot\nabla = 0 \tag{5}$$

Using this result, one can see that Eq. (3) is formally identical to Eq. (1) and, therefore, a particular solution of Eq. (3) may be written as

$$B/B_o = \rho/\rho_o \tag{6}$$

where subscript o designates some reference chamber condition to be defined below. Equation (5) may be used also to simplify the body force term in the momentum equation. The result can be expressed as

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla B^2/2 = -\nabla B^2/2 \tag{7}$$

Introducing next the quasi one-dimensional assumption, Eqs. (1) and (2) reduce, for steady flow, to

$$\rho u A = \dot{m} \tag{8}$$

$$\rho u(du/dx) + (dp/dx) + (1/\mu)B(dB/dx) = 0 \tag{9}$$

where $\dot{m}$ is the actual mass flow rate and A is the area. For isentropic flow,

$$dh = dp/\rho \tag{10}$$

where h is the specific enthalpy. Therefore, using Eqs. (6) and (10), Eq. (9) can be integrated directly. Carrying out the indicated integration, one obtains

$$(u^2/2) + h + (B^2/\mu\rho) = \text{Const} = IV/\dot{m} \quad (11)$$

where I is the current and V is the voltage. The reference conditions are now defined by the relations

$$h_o + B_o^2/\mu\rho_o = IV/\dot{m}$$

and

$$B_o = \int B dA/\pi r_a^2 \quad (12)$$

where r_a is the anode radius. Referring to Fig. 1 and letting[11]

$$B = B_\theta = \begin{cases} \mu I/2\pi r_c^2 r & r \le r_c \\ \mu I/2\pi r & r > r_c \end{cases} \quad (13)$$

where r_c is the cathode radius, one finds

$$B_o = (\mu I/\pi r_a^2)[r_c/3 + (r_a - r_c)] \quad (14)$$

Another possible definition for B_o is

$$B_o^2 = \int B^2 dA/\pi r_a^2 = \frac{1}{2}\left(\frac{\mu I}{\pi r_a}\right)^2 \left[\ln\frac{r_a}{r_c} + \frac{1}{4}\right]$$

Simple expressions for the performance parameters can be obtained if one assumes that the plasma behaves like a perfect gas. In this case, Eqs. (11) and (12) give

$$(u^2/2) + c_pT + B^2/\mu\rho = (u^2/2) + (a^2/\gamma - 1) + c^2 = (a_o^2/\gamma - 1) + c_o^2 \quad (15)$$

where

$$a = (\gamma p/\rho)^{1/2} \quad c = (B^2/\mu\rho)^{1/2} \quad (16)$$

are the speeds of sound and of an Alfvén wave, respectively, c_p is the specific heat at constant pressure, and γ is the effective ratio of specific heats. Letting

$$\xi = T/T_o,\ \lambda^2 = (\gamma - 1)c_o^2/a_o^2,\ M = u/a \quad (17)$$

Eq. (15) gives

$$u = a_o\{(2/\gamma - 1)[1 + \lambda^2 - \xi - \lambda^2\xi^{1/(\gamma-1)}]\}^{1/2} \quad (18)$$

and

$$M^2 = (2/\gamma - 1)[(1 + \lambda^2/\xi) - 1 - \lambda^2\xi^{(2-\gamma)/(\gamma-1)}] \quad (19)$$

From Eq. (6) and the isentropic relations, one finds

$$B/B_o = \rho/\rho_o = \xi^{1/(\gamma-1)},\ p/p_o = \xi^{\gamma/(\gamma-1)} \quad (20)$$

The magnitude of λ^2 relative to one determines the relative importance of the electromagnetic terms relative to the aerodynamic terms. To get a feel for its magnitude one finds that, for an anode diameter of 1 in.,

$$\lambda^2 \simeq 1.5(\gamma - 1/\gamma)I^2/p_o \quad (21)$$

where I is in KA and p_o in torr. Thus, if I = 10 KA, p_o 100 torr, $\lambda^2 \simeq 0.6$ for $\gamma = 1.67$.

The equations obtained here are valid downstream of the anode where no power is added to the plasma. An expression for the effective area of the magnetic nozzle in such a region can be obtained from the continuity equation which gives

$$\frac{\dot{m}}{A} = \rho u = \left(\frac{\gamma}{R}\right)^{1/2} \frac{p}{T^{1/2}} M = \left(\frac{\gamma}{R}\right)^{1/2} \frac{p_o}{T_o^{1/2}} M\xi^{(\gamma+1)/2(\gamma-1)} \quad (22)$$

where R is the gas constant. Equations (19) and (22) show that $\dot{m}/A$ has a maximum when $\xi = \eta$, where η satisfies the

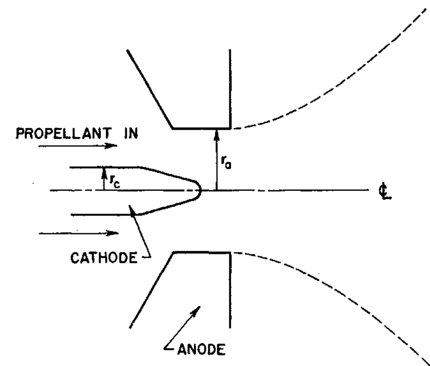

Fig. 1 Schematic of an MPD arc.

relation

$$3\lambda^2\eta^{1/(\gamma-1)} + (\gamma + 1)\eta - 2(1 + \lambda^2) = 0 \quad (23)$$

Rewriting Eq. (23) as

$$\lambda^2 = [(\gamma + 1)\eta - 2]/[2 - 3\eta^{1/(\gamma-1)}]$$

one finds

$$(\tfrac{2}{3})^{\gamma-1} \ge \eta \ge 2/\gamma + 1,\ \gamma < 2 \quad (24)$$

for all values of λ. Because $0.763 \ge \eta \ge 0.750$ for $\gamma = 1.67$ and $0.850 \ge \eta \ge 0.833$ for $\gamma = 1.4$, one can conclude that η is independent of λ. If the minimum area of the magnetic nozzle is the anode area, Fig. 1, then the maximum mass flow rate is given by

$$\dot{m} = (\gamma/A)^{1/2}(P_o/T_o^{1/2})\eta^{(\gamma+1)/2(\gamma-1)} \times \left[1 + \frac{\lambda^2}{\gamma - 1}\eta^{(2-\gamma)/(\gamma-1)}\right]^{1/2} A_a \quad (25)$$

and the Mach number at the anode exit is

$$M_a = [1 + (\lambda^2/\gamma - 1)\eta^{(2-\gamma)/(\gamma-1)}]^{1/2} > 1 \quad \text{when } \lambda \ne 0 \quad (26)$$

The desired equation for the shape of the magnetic nozzle follows from Eqs. (19, 22, and 25):

$$\frac{A}{A_a} = \frac{(\eta/\xi)^{(\gamma+1)/2(\gamma-1)}\left[1 + \dfrac{\lambda^2}{\gamma-1}\eta^{(2-\gamma)/(\gamma-1)}\right]^{1/2}}{\left\{\dfrac{2}{\gamma-1}\left[\dfrac{1+\lambda^2}{\xi} - 1 - \lambda^2\xi^{(2-\gamma)/(\gamma-1)}\right]\right\}^{1/2}} \quad (27)$$

The thrust is given by the relation,

$$F = \dot{m}u_e \quad (28)$$

where subscript e designates conditions in the exhaust region. Since ξ_e is expected to be small compared to unity, one obtains

$$u_e^2 = (F/\dot{m})^2 \simeq \frac{2}{\gamma - 1} a_o^2(1 + \lambda^2) = 2\frac{\gamma}{\gamma - 1} \times RT_o(1 + \lambda^2) \quad (29)$$

As can be seen from Eq. (21), λ^2 can be expressed as

$$\lambda^2 = bI^2/p_o \quad (30)$$

where b is approximately independant of mass flow rate and current, hence, Eq. (29) takes the form

$$u_e^2 = 2(\gamma/\gamma - 1)RT_o(1 + bI^2/p_o) \quad (31)$$

Results and Discussion

The pressure p_o appearing in the theory is purely aerodynamic and, as such, is approximately proportional to the

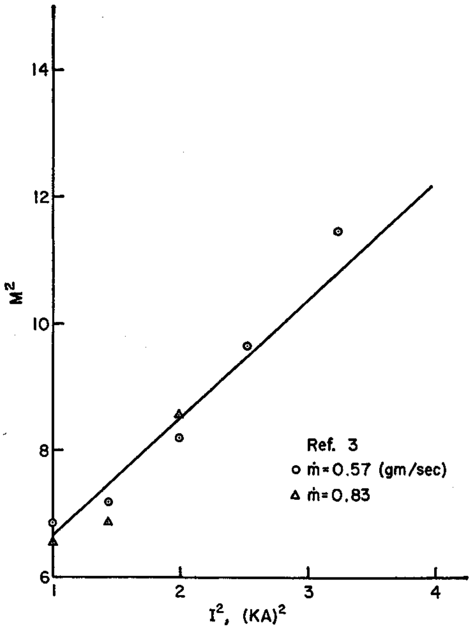

Fig. 2 Variation of Mach number with current.

actual mass flow rate. Experimentally measured chamber pressures include, on the other hand, both aerodynamic and electromagnetic components.[2,3] Equation (31) then indicates that, for a partially ionized gas and a given mass flow rate, the square of the exhaust velocity is approximately a linear function of the square of the current. Also, for a given current, the square of the velocity is a linear function of the inverse of the mass flow rate. Similar statements can be made regarding the exhaust Mach number. This, in turn, implies that the thrust is not the sum of the electromagnetic and aerodynamic components but depends on both in a nonlinear manner. The quantities T_o, γ, and b appearing in Eq. (31) have, in general, slight dependence on mass flow rate and current. For a given p_o, T_o and γ can be computed from Eq. (12), provided enough assumptions are made regarding the ionization process in the arc chamber. The details of the computations are similar to those described in Chapter 6 of Ref. 11 for electrothermal engines. Because, in partially ionized gases, most of the energy goes into ionization, T_o and γ are expected to remain fairly constant. On the other hand, when the gas is fully ionized, T_o will increase with I and the simple dependance, indicated by Eq. (31) when T_o and γ are constants, will not hold.

To compare the predictions of the theory with experiment one needs Mach number and velocity measurements. The

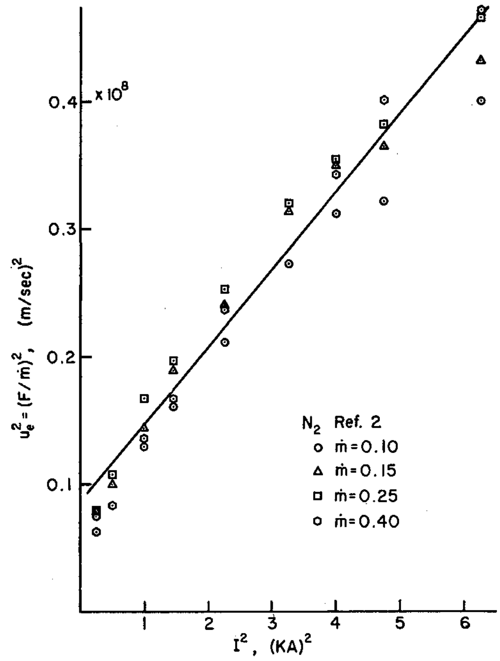

Fig. 4 Thrust over mass flow rate variation with current for N_2.

velocity computed in this theory is a mean average velocity and, because of the one-dimensional approximation, has no radial dependance. Also, for a meaningful comparison with experiment, the predictions of Eq. (31) should be compared with measurements of the fully developed velocity. Similar remarks apply to the Mach number.

Mach number measurements were reported in Refs. 3 and 7. The data of Ref. 3 were redrawn in Fig. 2, and it is seen that they are in good agreement with trends predicted by the theory. However, it was stressed in both Refs. 3 and 7 that the measurements were rather crude. Thus, Mach numbers were estimated from jet photographs in Ref. 3. On the other hand, wedges employed in Ref. 7 to measure the Mach numbers did not yield consistent results. Unless indicated otherwise, the solid lines in Fig. 2 and subsequent figures are best fits of data not deemed influenced by erosion and/or entrainment.

Velocity measurements were reported in Refs. 2, 4, 7, 8, and 9. The velocity in Refs. 2 and 4 was measured by induction velometry and estimated from measurements of momentum and mass fluxes; however, results based on the measurements of the fluxes were not deemed reliable. Time-of-flight probe

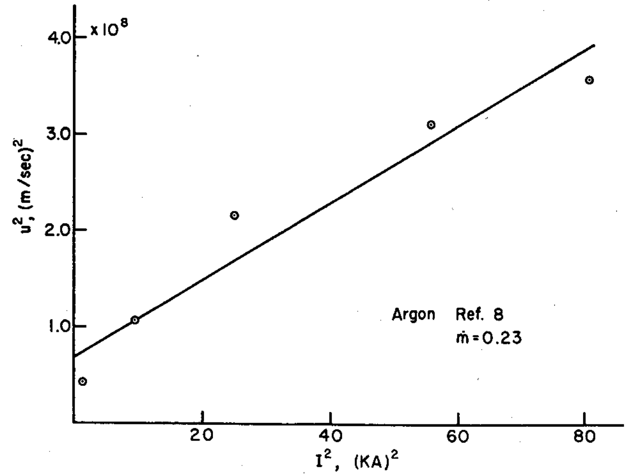

Fig. 3 Velocity vs current for argon.

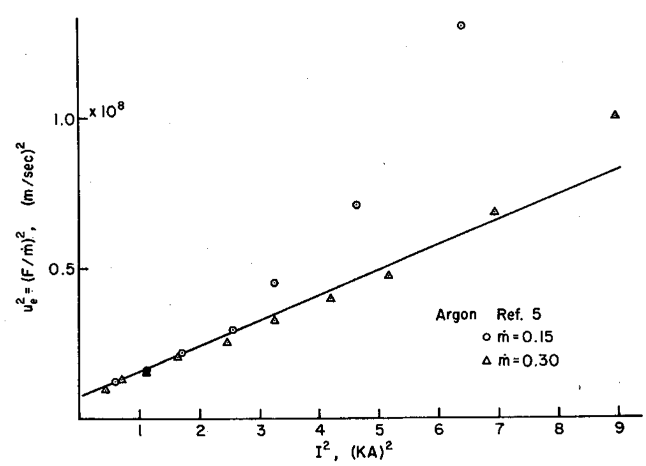

Fig. 5 Thrust over mass flow rate vs current for argon.[5]

techniques were employed in Refs. 7 and 8. Not enough data were presented in Refs. 2, 4, and 7 to prove or disprove the predicted dependance of velocity on current for a given mass flow rate. Plot of the average of the raw data of Ref. 8 for $\dot{m} = 0.23$ g/sec is shown in Fig. 3; because the velocities are not the fully developed velocities, no meaningful comparison with theory can be made. A Doppler shift method is employed in Ref. 9 to measure the velocity at a given current for various flow rates. The measurements indicate that the velocity increases with decreasing mass flow rate in agreement with the theory. In the judgment of the authors of Ref. 9, only measurements at the two highest mass flow rates were not affected by erosion, therefore, no attempt was made to plot their data in the form u_e^2 vs $1/\dot{m}$.

The measurements of Refs. 7 and 9 indicate that below a certain metered mass flow rate, the velocity reaches a plateau determined by the current. This result follows from Eq. (31) if one assumes that, for a given current, there is a minimum mass flow rate below which the actual mass flow rate remains constant as the metered mass flow rate is decreased. Because p_o is proportional to the mass flow rate, Eq. (31) can be used to suggest a criterion for electrode erosion and/or entrainment. As is seen from Eq. (31), a plot of u_e^2 vs $1/\dot{m}$ at a given current is a straight line; any departure from a straight line when u_e^2 is plotted versus the inverse of the metered mass flow rate indicates erosion and or entrainment. This scheme adds yet another method for estimating the "matched mass flow,"[7] which is the mass flow rate below which erosion and/or entrainment become appreciable.

If measurements of thrust, mass flow rate and current are available, then this theory presents a second method for predicting erosion and/or entrainment, provided gas is partially ionized. This may be achieved by plotting $(F/\dot{m})^2$ vs I^2; departure from a straight line indicates erosion and/or entrainment. The measured value of thrust, if reliable, can be used to estimate the additional mass flow rate by computing the difference between the theoretically predicted value and the metered value. A plot of the data of Refs. 2 and 4 for N_2 is shown in Fig. 4, the data of Ref. 5 for argon is shown in Fig. 5, whereas the data of Ref. 8 for argon is shown in Fig. 6 in the form I_{sp}^2 vs I^2 where $I_{sp} = F/\dot{m}g_o$, g_o being the gravity at sea level. The data points in Figs. 4 and 6 represent the average of the raw data. At the higher mass flow rates, the data exhibits the dependance predicted by the theory. However, departures are noted at low mass flow rates and high currents. Based on this criterion, the influence of erosion and/or entrainment on the data of Refs. 2 and 4 is insignificant. Referring to Fig. 6, it is seen that a straight line gives a good fit when $\dot{m} = 0.23$. On the other hand when $\dot{m} = 0.17$ departure from a straight line takes place at about 7000A. This implies, according to this criterion, that erosion takes place at a current of 7000A when $0.17 < \dot{m} < 0.23$ g/sec. The criterion of Ref. 7 gives a matched mass flow rate of 0.93 g/sec at this current. If this procedure for predicting erosion and/or entrainment is valid then one can see by comparing Figs. 5 and 6 that the mass flow rates below which erosion and/or entrainment become important at a given current have strong dependance on the device and its environment. Thus, the cirteria for erosion and/or entrainment discussed in Ref. 9 are not expected to have general validity.

If the gas is fully ionized, the erosion criterion based on thrust measurements is not expected to hold. In this case T_o will increase with I; thus, u_e^2 will increase with I at a rate faster than that indicated by Eq. (31) when T_o is approximately constant.

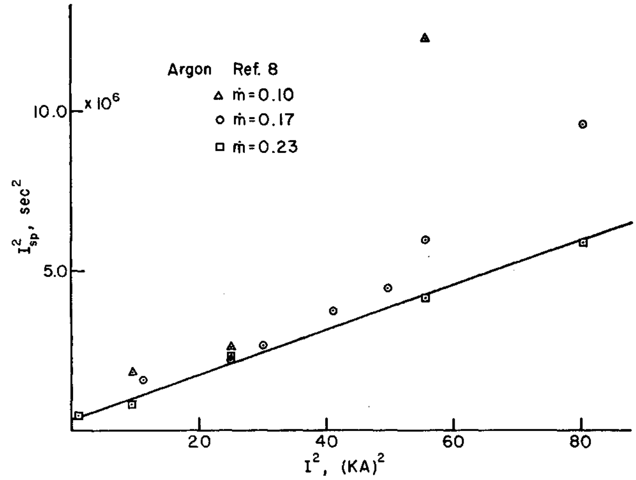

Fig. 6 Variation of specific impulse with current for argon.[8]

Concluding Remarks

Because of the nonlinear interaction of the current and flowfield, the resulting expression for the thrust is such that it is not possible to write it as the sum of the aerodynamic and electromagnetic components. The theory predicts that the square of the exhaust velocity, at a given current, is approximately a linear function of the inverse of the mass flow rate. If the gas is partially ionized, the theory also predicts that the square of the ratio of thrust to mass flow rate is approximately a linear function of the square of the current for a given mass flow rate. It is suggested that departure from such dependance indicates onset of erosion and/or entrainment.

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